Moriah College

בית ספר הר המוריה

2021 HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

Reading time – 10 minutes

- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7–10)

- Attempt Questions 11–14
- Allow about 1 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

---

1 An examination consists of 20 questions, each having four possible answers. A student guesses the answer to every question. Let X be the number of correct answers. What is the value of $E(X)$?

A. 4

B. 5

C. 10

D. 16

2 Evaluate $|\underset{\sim}{a}+\underset{\sim}{b}|.\underset{\sim}{c}$ given that $\underset{\sim}{a}=\begin{pmatrix}1\\2\end{pmatrix}$, $\underset{\sim}{b}=\begin{pmatrix}3\\1\end{pmatrix}$ and $\underset{\sim}{c}=\begin{pmatrix}6\\-2\end{pmatrix}$.

A. 18

B. $\begin{pmatrix}30\\-10\end{pmatrix}$

C. 20

D. $5\sqrt{40}$

3 Given that $h(x)=\sqrt{2-x}$, what are the domain and range of $h^{-1}(x)$?

A. $x\le 2,\quad y\ge 0$

B. $x\le 2,\quad y\le 0$

C. $x\ge 0,\quad y\le 2$

D. $x\ge 0,\quad y\le -2$

4 Which of the following is a primitive of $\frac{4}{\sqrt{1-4x^2}}$?

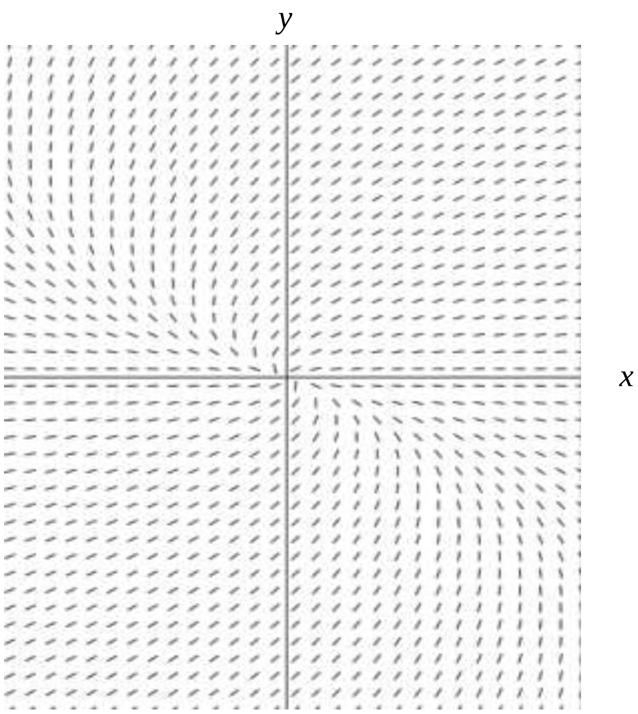
A. $2\sin^{-1}(2x)+C$

B. $2\sin^{-1}\left(\frac{x}{2}\right)+C$

C. $\sin^{-1}\frac{x}{2}+C$

D. $\sin^{-1}(2x)+C$

5 The slope field for a differential equation is shown below



Which of the following could be the differential equation represented?

A. $\frac{dy}{dx}=\frac{x}{x+y}$

B. $\frac{dy}{dx}=\frac{y}{x+y}$

C. $\frac{dy}{dx}=\frac{x}{x-y}$

D. $\frac{dy}{dx}=\frac{y}{x-y}$

6 Three candidates are running for election. There are 202 people who vote. The winner is the person who gets the most votes. What is the minimum number of votes needed for someone to win the election?

A. 101

B. 102

C. 67

D. 68

7 Which expression would be the correct answer to $\int \sin(5x)\sin(3x)\,dx$?

A. $\frac{1}{4}\cos(2x)+\frac{1}{16}\cos(8x)+C$

B. $\frac{1}{4}\cos(2x)-\frac{1}{16}\cos(8x)+C$

C. $\frac{1}{4}\sin(2x)+\frac{1}{16}\sin(8x)+C$

D. $\frac{1}{4}\sin(2x)-\frac{1}{16}\sin(8x)+C$

8 A team of 7 is to be selected from 10 people and then sat around a circular table.
In how many ways can this be done?

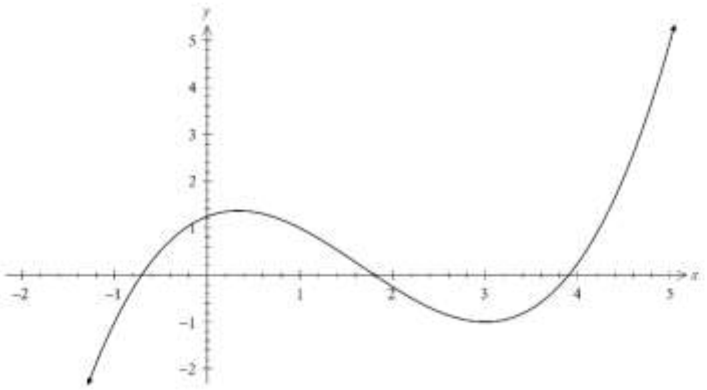
A. $\frac{10!}{7!3!}\times 7!$

B. $\frac{10!}{7!3!}\times\frac{1}{5!}$

C. $\frac{10!}{7\times 3!}$

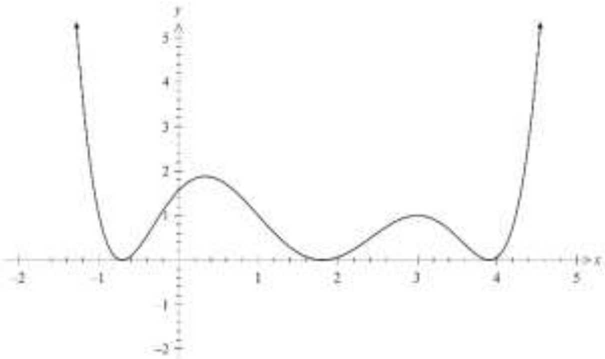
D. $\frac{10!}{7!3!}$

9 The graph of the function $y = f(x)$ is drawn below

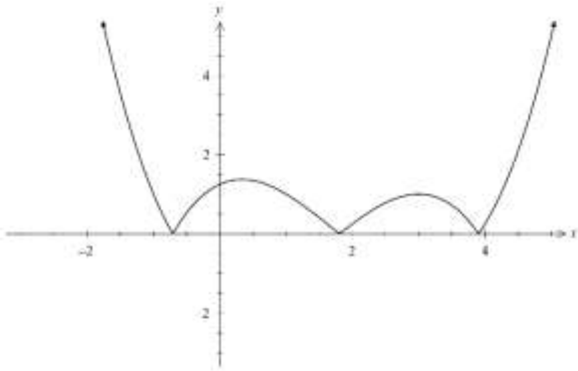


Which of the following represents the graph of $y = |f(|x|)|$?

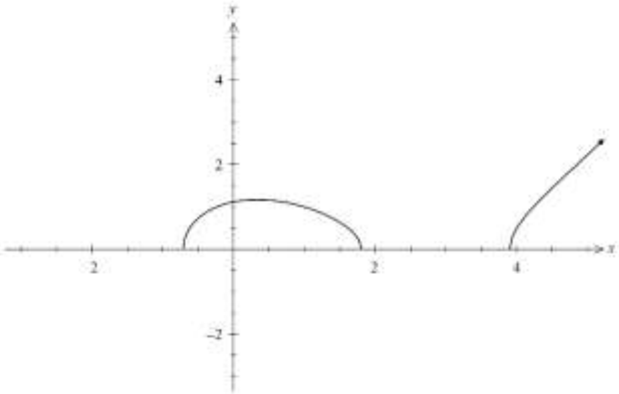
A



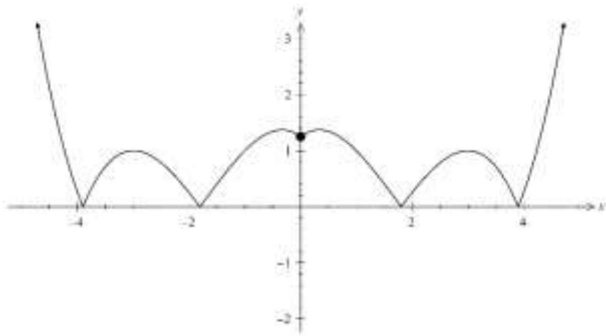
B



C



D



10 The parametric equations of a curve C are given by $x = \operatorname{cosec}^2 t$ and $y = \cos^2 t$.
What is the Cartesian equation of C ?

A. $y = 1 - \dfrac{1}{x}$

B. $y = 1 + \dfrac{1}{x}$

C. $xy = 1$

D. $y^2 = 1 - \dfrac{1}{x^2}$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question on a separate page
For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

(a) $P(x) = 2x^3 + 11x^2 + 12x - 9$

(i) Show that $P(-3) = 0$ **1**

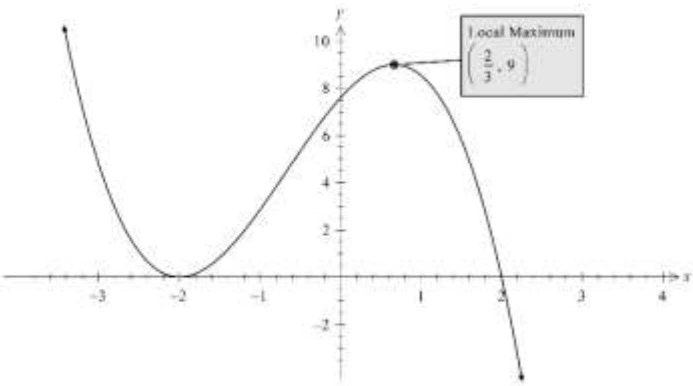
(ii) Show that $P'(-3) = 0$ **1**

(iii) Hence express $2x^3 + 11x^2 + 12x - 9$ as the product of 3 linear factors. **1**

(b) For what value(s) of a are the two distinct vectors $\begin{pmatrix} a \\ 2a+6 \end{pmatrix}$ and $\begin{pmatrix} a+1 \\ -1 \end{pmatrix}$ perpendicular? **2**

(c) The diagram shows the sketch of $y = f(x)$. Sketch the graph of $y^2 = f(x)$. **3**

Indicate the local maximum turning point on your diagram.



(d) By letting $t = \tan\dfrac{x}{2}$, solve $4\sin x - 3\cos x = 3$, for $0 \le x \le 2\pi$ **3**

(e) Solve $2\sin^2 2x - 1 = 0$, over the domain $[0, \pi]$. **2**

(f) A function is defined by $f(x) = \sin^{-1}(1-x)$.

(i) Find the domain of $f(x)$. **1**

(ii) Draw a neat sketch of $y = f(x)$. **1**

Question 12 (15marks)

(a) Use mathematical induction to prove that $n^3 + 2n$ is divisible by 3, for all positive integers *n*. **3**

(b) Weather observations in the town of Dampville have established that the probability of rain on any given day is 0.8.

Observations are made for 100 consecutive days.

Let *X* be the random variable representing the number of rainy days

(i) Find the expected value $E(X)$. **1**

(ii) Find the standard deviation of *X*. **1**

(iii) By considering a normal distribution find the approximate probability that $76 \le X \le 84$. **1**

(c) Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 2x\, dx$. **3**

(d) Solve the differential equation $\frac{dy}{dx} = \frac{-x}{1+y^2}$, given that $y(-1) = 1$. **3**

Express your answer in the form $ay^3 + bx^2 + cy + d = 0$, where *a*,*b*,*c*,*d* are integers.

(e) The roots of $x^3 - 2x^2 + 5x + 1 = 0$ are α, β and γ. Find the value of

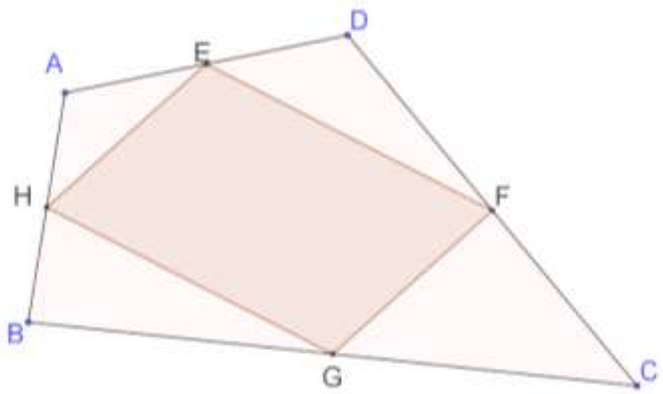
(i) $5\alpha + 5\beta + 5\gamma$. **1**

(ii) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. **1**

(iii) $\alpha^2 + \beta^2 + \gamma^2$. **1**

Question 13 (15marks)

(a) Varignon's theorem tells us that if you join the midpoints of the sides of any quadrilateral, the resulting quadrilateral will be a parallelogram.



E, *F*, *G* and *H* are the midpoints of *AD*, *DC*, *CB* and *BA* respectively.
Let $\overrightarrow{BA} = \underset{\sim}{u}, \overrightarrow{AD} = \underset{\sim}{v}, \overrightarrow{DC} = \underset{\sim}{w}$ and $\overrightarrow{CB} = \underset{\sim}{z}$

(i) Explain why $\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w} + \underset{\sim}{z} = 0$ **1**

(ii) Show that $\overrightarrow{HE} = \frac{1}{2}(\underset{\sim}{u} + \underset{\sim}{v})$ **1**

(iii) Show that $\overrightarrow{GF} = -\frac{1}{2}(\underset{\sim}{w} + \underset{\sim}{z})$ **1**

(iv) Use your result in (i) to explain why *HE* is parallel to *GF* **2**

(b) Use the substitution $u = 3 + e^x$ to find the exact value of $\int_0^{\ln 6} \frac{e^x}{\sqrt{3+e^x}}\,dx$. **3**

(c) A population of monkeys on a small island is an example of logistic growth.
The population of the monkeys *P* is given $P = \frac{50}{1+24e^{-2t}}$ and *t* is in years.

(i) Determine the carrying capacity of the population. That is the value of *P* as $t \to \infty$ **1**

(ii) Show that $\frac{dP}{dt} = \frac{1}{25}P(50 - P)$. **4**

(iii) Draw a neat sketch of $P = \frac{50}{1+24e^{-2t}}$, taking care to consider the shape of the function at $t = 0$. **2**

Question 14 (15 marks)

(a) Find the value of n if ${}^nC_2 + {}^nC_1 + {}^nC_0 = 172$ **3**

(b) Find the volume of the solid of revolution formed when the graph of $y = \sqrt{\dfrac{1+2x}{1+x^2}}$ **3**

is rotated around the x – axis over the interval [0,1]

(c) Spherical balloons are being inflated for a party. Empty balloons are being inflated so that the volume increases at a rate of $8\,cm^3/\text{s}$.

(i) By finding $\frac{dr}{dt}$ in terms of r, or otherwise,

show that the radius at time t is given by $r = \sqrt[3]{\dfrac{6t}{\pi}}$. **3**

(ii) Find the rate of increase of the surface area after 4 seconds. **3**

(iii) The balloon will burst when the surface area reaches $3000\,cm^2$. **3**
After how many seconds should inflation stop?

Teacher Training Australia

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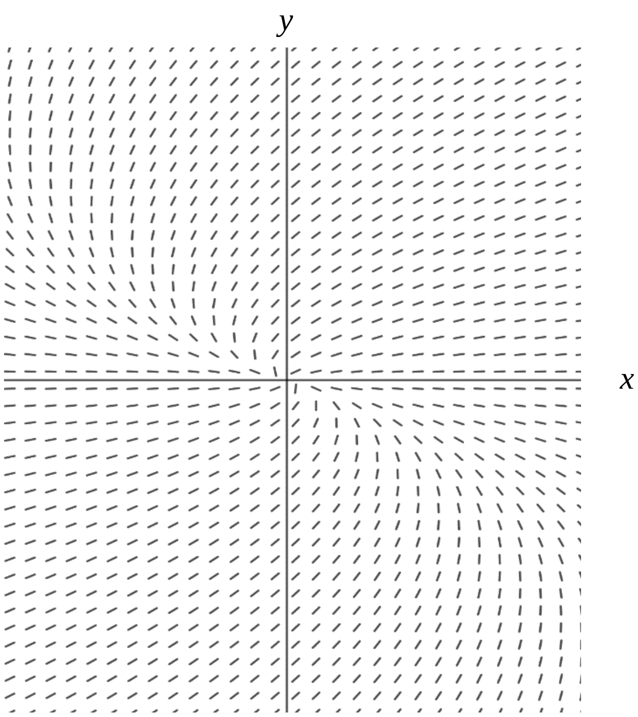

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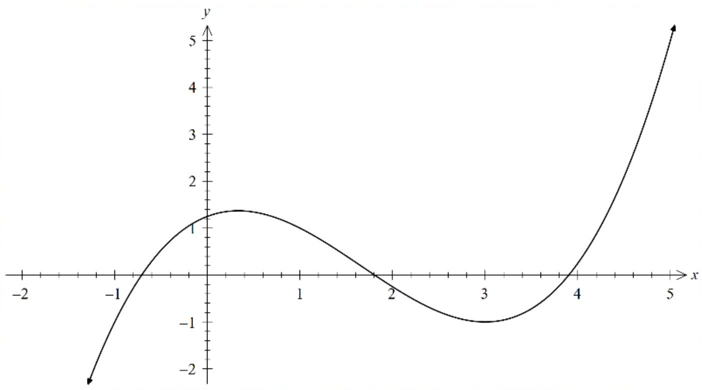

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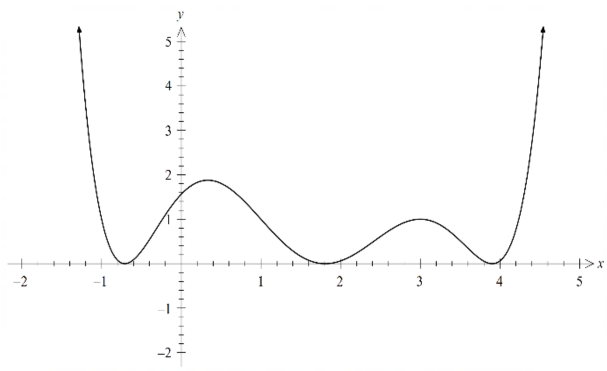
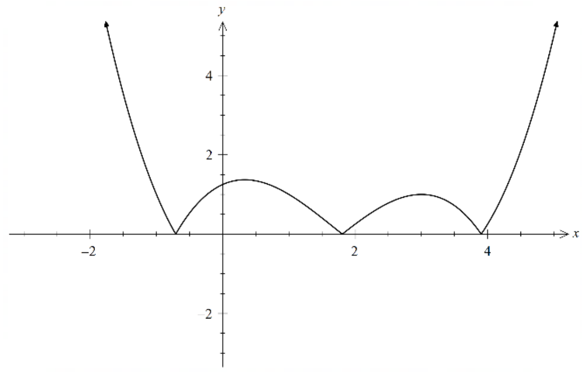
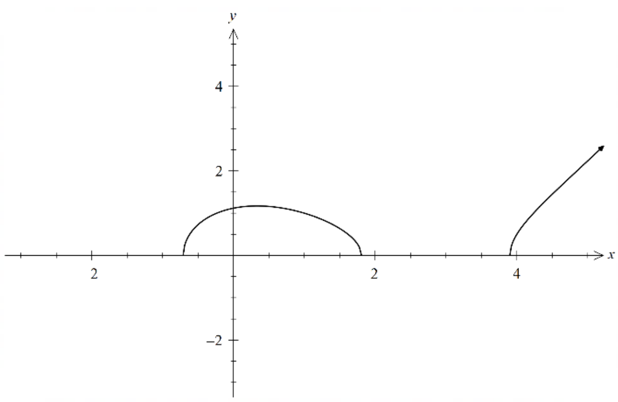


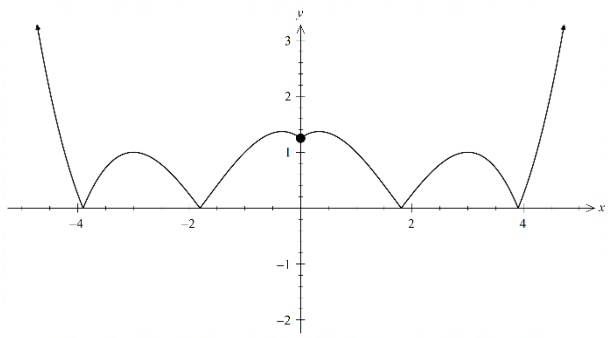
B



C



D



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Section II

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(a) $P(x) = 2x^3 + 11x^2 + 12x - 9$

(i) Show that $P(-3) = 0$ **1**

$$P(x) = 2x^3 + 11x^2 + 12x - 9$$
$$P(-3) = 2(-3)^3 + 11(-3)^2 + 12(-3) - 9$$
$$= -54 + 99 - 36 - 9$$
$$= 0 \quad \checkmark$$

(ii) Show that $P'(-3) = 0$ **1**

$$P(x) = 2x^3 + 11x^2 + 12x - 9$$
$$P'(x) = 6x^2 + 22x + 12$$
$$P'(-3) = 6(-3)^2 + 22(-3) + 12$$
$$= 54 - 66 + 12 = 0 \quad \checkmark$$

(iii) Hence express $2x^3 + 11x^2 + 12x - 9$ as the product of 3 linear factors. **1**

$$P(x) = (x+3)^2(2x-1) \quad \checkmark$$

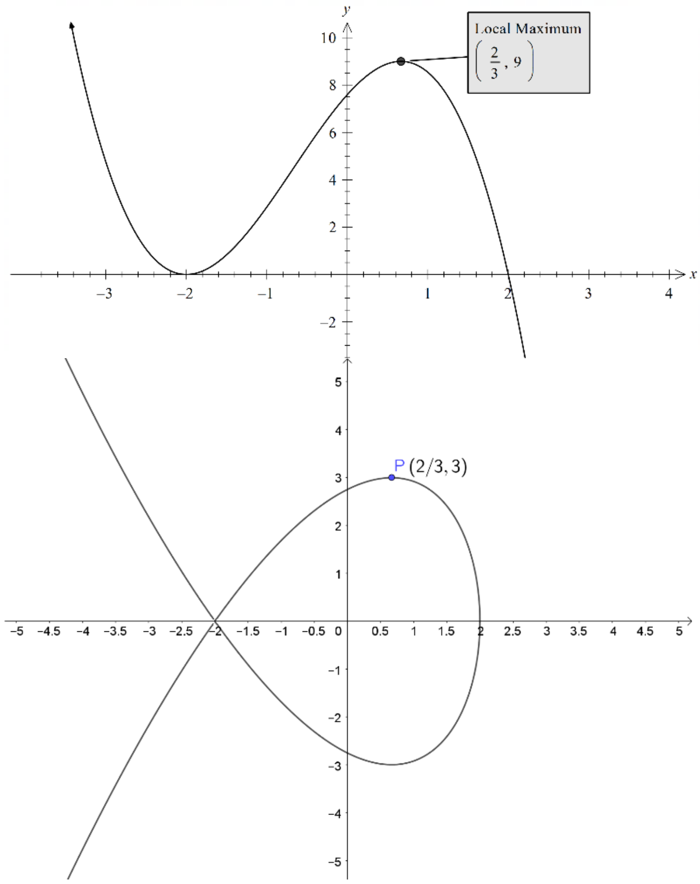
(b) For what value(s) of a are the two distinct vectors $\begin{pmatrix} a \\ 2a+6 \end{pmatrix}$ and $\begin{pmatrix} a+1 \\ -1 \end{pmatrix}$ perpendicular? **2**

To be perpendicular the dot product of the vectors must be 0. ✓

$$\begin{pmatrix} a \\ 2a+6 \end{pmatrix} . \begin{pmatrix} a+1 \\ -1 \end{pmatrix} = a^2 + a - 2a - 6$$
$$= a^2 - a - 6$$
$$a^2 - a - 6 = 0$$
$$(a-3)(a+2) = 0 \rightarrow a = 3, -2 \quad \checkmark$$

(c) The diagram shows the sketch of $y = f(x)$. Sketch the graph of $y^2 = f(x)$. **3**

Indicate the local maximum turning point on your diagram.



✓✓✓

(d) By letting $t = \tan\frac{x}{2}$, solve $4\sin x - 3\cos x = 3, \text{for } 0 \le x \le 2\pi$ **3**

$$\text{Let } t = \tan\frac{x}{2} \rightarrow 4 \times \frac{2t}{1+t^2} - 3 \times \frac{1-t^2}{1+t^2} = 3 \quad ✓$$

$$\rightarrow 8t - 3 + 3t^2 = 3 + 3t^2$$

$$t = \frac{3}{4} \rightarrow x = 2\arctan\left(\frac{3}{4}\right) = 1.287\,(3dp) \quad ✓$$

Check $x = \pi \rightarrow 4\sin\pi - 3\cos\pi = 0 - -3 = 3$ ✓

$\therefore x = \pi$ is also a solution.

(e) Solve $2\sin^2 2x - 1 = 0$, over the domain $[0, \pi]$ finding x as a function of y. **2**

$$2\sin^2 2x - 1 = 0$$
$$\sin^2 2x = \frac{1}{2}$$
$$\sin 2x = \pm\frac{1}{\sqrt{2}} \quad 0 \le 2x \le 2\pi \quad ✓$$
$$2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$
$$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8} \quad ✓$$

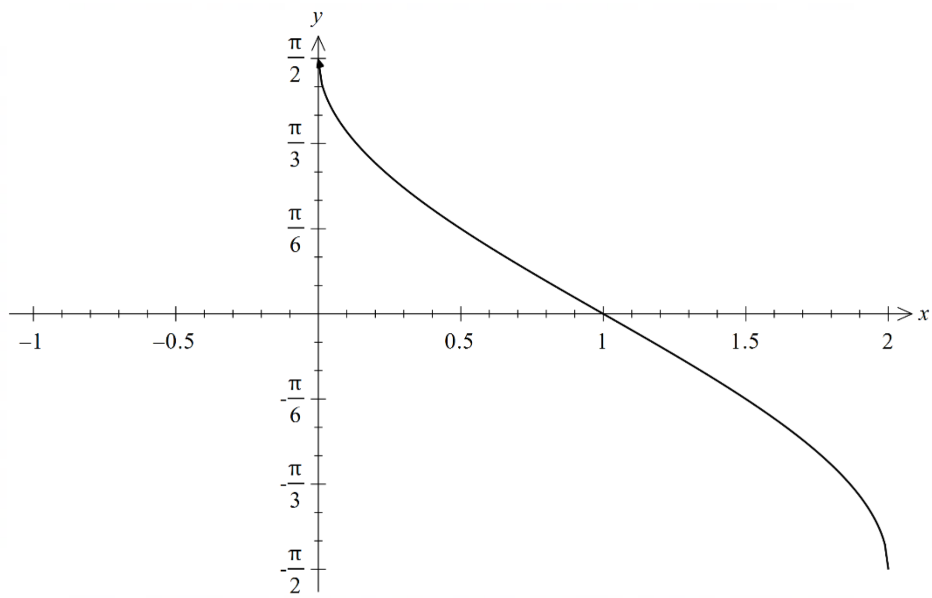
(f) A function is defined by $f(x) = \sin^{-1}(1 - x)$.

(i) Find the domain of $f(x)$. **1**

$$-1 \le 1 - x \le 1$$
$$-2 \le -x \le 0$$
$$0 \le x \le 2 \quad ✓$$

(ii) Draw a neat sketch of $y = f(x)$. **1**

✓



Question 12 (15marks)

(a) Use mathematical induction to prove that $n^3 + 2n$ is divisible by 3, for all positive integers n. **3**

Prove true for $n = 1$

$1^3 + 2\times1 = 3$ which is divisible by 3 so true for $n = 1$ ☑

Assume true for $n = k$

$\frac{k^3+2k}{3} = M$ (where M is a positive integer)

$k^3 = 3M - 2k$

Prove true for $n = k + 1$

RTP that $(k+1)^3 + 2(k+1)$ is divisible by 3

$(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$

$= k^3 + 3k^2 + 5k + 3$

$= 3M - 2k + 3k^2 + 5k + 3$ (from assumption) ☑

$= 3(M + K + 1) \rightarrow$ (divisible by 3) ☑

true for $n = k + 1$, proved by mathematical induction

(b) Weather observations in the town of Dampville have established that the probability of rain on any given day is 0.8.

Observations are made for 100 consecutive days.

Let X be the random variable representing the number of rainy days.

(i) Find the expected value $E(X)$. **1**

$E(X) = np = 100\times0.8 = 80$ days ✓

(ii) Find the standard deviation of X. **1**

$\sigma = \sqrt{npq} = \sqrt{100\times0.8\times0.2} = 4$ ✓

(iii) By considering a normal distribution find the approximate probability that $76 \le X \le 84$. **1**

This represents one standard deviation from the mean so

$P(76 \le X \le 84) \approx 0.68$ ✓

(c) Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 2x\,dx$ **3**

$$\int_0^{\frac{\pi}{2}} \cos^2 2x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 4x)\,dx \quad ✓$$

$$= \frac{1}{2}\left[x + \frac{\sin(4x)}{2}\right]_0^{\frac{\pi}{2}} \quad ✓$$

$$= \frac{1}{2}\left(\frac{\pi}{2} + \frac{\sin\left(4\times\frac{\pi}{2}\right)}{2} - 0\right)$$

$$= \frac{\pi}{4} \quad ✓$$

(d) Solve the differential equation $\frac{dy}{dx} = \frac{-x}{1+y^2}$, given that $y(-1)=1$ **3**

Express your answer in the form $ay^3 + bx^2 + cy + d = 0$, where a,b,c,d are integers.

$$\frac{dy}{dx} = \frac{-x}{1+y^2}$$

$$dy\left(1+y^2\right) = -x\,dx$$

$$y + \frac{y^3}{3} = -\frac{x^2}{2} + C \text{ and } y(-1) = 1 \; ☑$$

$$1 + \frac{1^3}{3} = -\frac{(-1)^2}{2} + C \to C = \frac{11}{6} \; ☑$$

$$y + \frac{y^3}{3} = -\frac{x^2}{2} + \frac{11}{6} \to 2y^3 + 3x^2 + 6y - 11 = 0 \; ☑$$

(e) The roots of $x^3 - 2x^2 + 5x + 1 = 0$ are α, β and γ. Find the value of

(i) $5\alpha + 5\beta + 5\gamma$. **1**

$$5\alpha + 5\beta + 5\gamma = 5(\alpha + \beta + \gamma)$$

$$= 5 \times 2 = 10 \; ☑$$

(ii) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. **1**

$$\alpha\beta\gamma(\alpha + \beta + \gamma) = -1 \times 2 = -2 \; ☑$$

(iii) $\alpha^2 + \beta^2 + \gamma^2$. **1**

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

$$= 2^2 - 2(5) = -6 \; ☑$$

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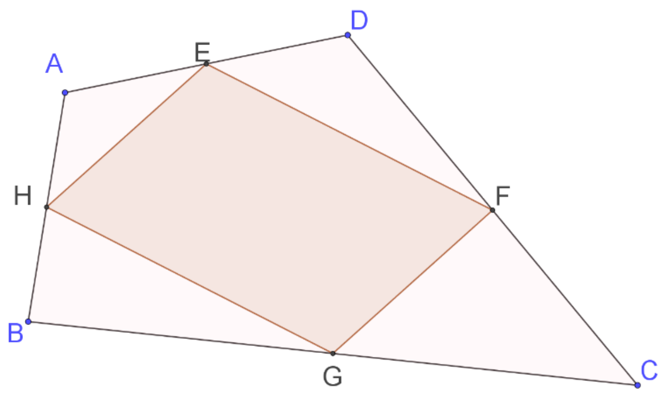
(a) Varignon's theorem tells us that if you join the midpoints of the sides of any quadrilateral, the resulting quadrilateral will be a parallelogram.



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(i) Explain why $\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w} + \underset{\sim}{z} = 0$ **1**

$\overrightarrow{BA} + \overrightarrow{AD} + \overrightarrow{DC} + \overrightarrow{CB} = \overrightarrow{BB} = 0$ ✓

(ii) Show that $\overrightarrow{HE} = \frac{1}{2}(\underset{\sim}{u} + \underset{\sim}{v})$ **1**

$\overrightarrow{HA} + \overrightarrow{AE} = \overrightarrow{HE}$

$\overrightarrow{HE} = \frac{1}{2}\underset{\sim}{u} + \frac{1}{2}\underset{\sim}{v}$

$\overrightarrow{HE} = \frac{1}{2}(\underset{\sim}{u} + \underset{\sim}{v})$ ✓

(iii) Show that $\overrightarrow{GF} = -\frac{1}{2}(\underset{\sim}{w} + \underset{\sim}{z})$ **1**

$\overrightarrow{GF} = \overrightarrow{GC} + \overrightarrow{CF}$

$= -\frac{1}{2}\underset{\sim}{z} + -\frac{1}{2}\underset{\sim}{w}$

$= -\frac{1}{2}(\underset{\sim}{w} + \underset{\sim}{z})$ ✓

(iv) Use your result in (i) to explain why *HE* is parallel to *GF* **2**

$\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w} + \underset{\sim}{z} = 0$

$\underset{\sim}{u} + \underset{\sim}{v} = -(\underset{\sim}{w} + \underset{\sim}{z})$ ✓

$\overrightarrow{HE} = \overrightarrow{GF}$

Since the vectors are equal they are parallel. ✓

(b) Use the substitution $u = 3 + e^x$ to find the exact value of $\int_0^{\ln 6} \frac{e^x}{\sqrt{3+e^x}}\,dx$ **3**

$u = 3 + e^x \to du = e^x dx$ and $u_1 = 3 + e^{\ln 6} = 9$ and $u_2 = 3 + e^0 = 4$ ☑

$$\int_0^{\ln 6} \frac{e^x}{\sqrt{3+e^x}}\,dx = \int_4^9 \frac{du}{\sqrt{u}} \quad ✓$$

$$= 2\left[\sqrt{u}\right]_4^9$$

$$= 2\left(\sqrt{9} - \sqrt{4}\right) = 2 \quad ✓$$

(c) A population of monkeys on a small island is an example of logistic growth. The population of the monkeys P is given $P = \frac{50}{1+24e^{-2t}}$ and t is in years.

(i) Determine the carrying capacity of the population. **1**

As $t \to \infty, P \to \frac{50}{1+0}$ so the carrying capacity is 50 monkeys. ✓

(ii) Show that $\frac{dP}{dt} = \frac{1}{25}P(50-P)$. **4**

$$P = \frac{50}{1+24e^{-2t}} = 50\left(1+24e^{-2t}\right)^{-1}$$

$$\frac{dP}{dt} = -50\left(1+24e^{-2t}\right)^{-2} \times -48e^{-2t} \quad ✓$$

$$= \frac{2400e^{-2t}}{\left(1+24e^{-2t}\right)^2} \text{ and } P = \frac{50}{1+24e^{-2t}} \to 1+24e^{-2t} = \frac{50}{P} \quad ✓$$

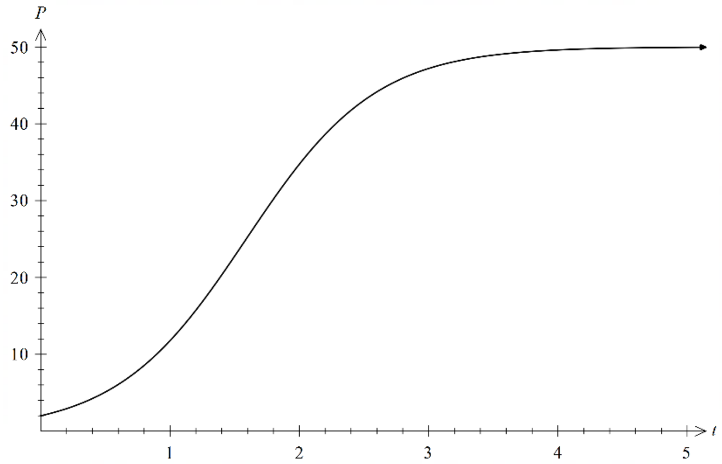
$$= \frac{24e^{-2t} \times 100}{\left(\frac{50}{P}\right)^2}$$

$$= \frac{100 \times \left(\frac{50}{P} - 1\right)}{\left(\frac{50}{P}\right)^2} ✓$$

$$= 100 \times \left(\frac{50}{P} - 1\right) \times \frac{P^2}{2500}$$

$$= \frac{1}{25}P(50-P) \quad ✓$$

(iii) Draw a neat sketch of $P = \dfrac{50}{1+24e^{-2t}}$, taking care to consider the shape of the function at $t = 0$. **2**



✓ ✓

Question 14 (15 marks)

(a) Find the value of n if ${}^nC_2 + {}^nC_1 + {}^nC_0 = 172$ **3**

$${}^nC_2 + {}^nC_1 + {}^nC_0 = 172$$

$$\frac{n(n-1)}{2} + n + 1 = 172 \quad ✓$$

$$n^2 - n + 2n + 2 = 344$$

$$n^2 + n - 342 = 0 \quad ✓$$

$$(n-18)(n+19) = 0$$

$$n = 18, n \neq -19 ✓$$

(b) Find the volume of the solid of revolution formed when the graph of $y = \sqrt{\dfrac{1+2x}{1+x^2}}$ is rotated around the x – axis over the interval [0,1] **3**

$$V = \pi \int_0^1 \left(\sqrt{\frac{1+2x}{1+x^2}} \right)^2 dx$$

$$V = \pi \int_0^1 \left(\frac{1+2x}{1+x^2} \right) dx \quad \boxed{✓}$$

$$V = \pi \int_0^1 \left(\frac{1}{1+x^2} + \frac{2x}{1+x^2} \right) dx$$

$$V = \pi \left[\arctan x + \ln\left(1+x^2\right) \right]_0^1 \quad \boxed{✓}$$

$$V = \pi \left(\left[\arctan 1 + \ln\left(1+1^2\right) \right] - \left(\arctan 0 + \ln 1 \right) \right)$$

$$V = \pi \left(\frac{\pi}{4} + \ln 2 \right) u^3 \quad \boxed{✓}$$

(c) Let $f(x)=\dfrac{1}{\arcsin x}$. **3**

Find $f'(x)$ and the largest possible for which $f'(x)$ is defined.

$$f(x)=\frac{1}{\arcsin x}=\left(\arcsin x\right)^{-1}$$

$$f'(x)=-\left(\arcsin x\right)^{-2}\times\frac{1}{\sqrt{1-x^2}} \quad \checkmark$$

$$=\frac{-1}{\left(\arcsin x\right)^2\sqrt{1-x^2}} \quad \checkmark$$

Both functions in the denominator have the same domain,but cannot be zero

So the domain is given by $-1<x<1$ ✓

(d) Spherical balloons are being inflated for a party. Empty balloons are being inflated so that the volume increases at a rate of $8\,cm^3/s$.

(i) Show that the radius at time t is given by $r=\sqrt[3]{\dfrac{6t}{\pi}}$. **2**

$$\frac{dV}{dt}=8 \text{ and } \frac{dV}{dr}=4\pi r^2$$

$$\frac{dr}{dt}=\frac{dr}{dV}\times\frac{dV}{dt}$$

$$=\frac{1}{4\pi r^2}\times 8$$

$$=\frac{2}{\pi r^2} \quad \checkmark$$

$$\frac{dt}{dr}=\frac{\pi r^2}{2}$$

$$t=\frac{\pi r^3}{6}+C, but\ when\ t=0, r=0 \therefore C=0$$

$$r^3=\frac{6t}{\pi}\rightarrow r=\sqrt[3]{\frac{6t}{\pi}}\ cm \quad \checkmark$$

(ii) Find the rate of increase of the surface area after 4 seconds. **2**

$$S = 4\pi r^2$$

$$\frac{dS}{dr} = 8\pi r$$

$$\frac{ds}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$$

$$= 8\pi r \times \frac{2}{\pi r^2}$$

$$= \frac{16}{r} \quad \checkmark$$

$$= \frac{16}{\sqrt[3]{\frac{6\times 4}{\pi}}}$$

$$= 8.124\, cm^2 / \sec \quad \checkmark$$

(iii) The balloon will burst when the surface area reaches 3000 cm^2. **2**
After how many seconds should inflation stop?

$$S = 4\pi r^2$$

$$4\pi r^2 = 3000$$

$$r^2 = \frac{750}{\pi}$$

$$r = \sqrt{\frac{750}{\pi}}$$

$$\sqrt[3]{\frac{6t}{\pi}} = \sqrt{\frac{750}{\pi}} \quad \checkmark$$

$$\frac{6t}{\pi} = \left(\sqrt{\frac{750}{\pi}}\right)^3$$

$$t = \frac{\pi}{6}\left(\sqrt{\frac{750}{\pi}}\right)^3$$

$$= 1931 \sec\ (\text{nearest sec}) \quad \checkmark$$